Lower Bounds for Admissible k-tuples

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Abstract

We will upper bound the size of admissible set with a given width. This gives dually a lower bound for the minimal width of an admissible set given the set size.

1 Preliminaries

Definition 1.1. *Denote by M_d the maximal number of admissible numbers in an interval of length d (i.e., $[x, x+d-1]$ for some integer x).*

Definition 1.2. *Denote by $h_k = \min\{\max(H) - \min(H) : H$ is admissible and $|H| = k\}$*

It is immediate that

Claim 1.3. *For any $d_1, d_2 \in \mathbb{N}$, we have $M_{d_1+d_2} \le M_{d_1} + M_{d_2}$*

We use the following simple relation between h_k and M_d

Claim 1.4. *$h_k \le d-1$ iff $M_d \ge k$. Alternatively, $h_k \ge d$ iff $M_d \le k-1$.*

Now we discuss how to derive lower bounds on h_k. We start by an example how to derive a lower bound on h_{672} from Engelsma's table. To derive a lower bound for h_{672} we can use the fact that $h_{337} = 2270$ which means that $M_{2270} \le 336$ by Claim 1.4. Similarly, $M_{2286} \le 335$. Using Claim 1.3, we get $M_{2270+2286} \le 335 + 336$ and using Claim 1.4 we get $h_{672} \ge 2270 + 2286 = 4556$.

We can generalize this approach. First given Engelsma's table and Claim 1.4, we derive M_d for all $d \le 2328$, this are the accurate values. Then we use Claim 1.3 (and possibly dynamic programming) to get upper bounds on M_d for values greater than 2329. Then we use Claim 1.4 again to derive lower bounds on h_k.

2 Algorithmic Upper Bounds on M_d

Let $I = [x, x+d-1]$ be an interval of size d, and let P be a set of prime numbers. We will show a upper bound on the maximal admissible set in this interval. Let A be an optimal such set. And let $\{a_p\}_{p\in P}$ be the values which A avoids modulo p (if there are several such values pick one of them arbitrarily). We denote by

$$B_p^{a_p} = \{i \in I \;:\; i \equiv a_p (mod p)\}\ .$$

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Then $A = I - \bigcup_{p\in P} B_p^{a_p}$, and $|A| = |I| - |\bigcup_{p\in P} B_p^{a_p}|$. A lower bound on $|\bigcup_{p\in P} B_p^{a_p}|$ gives an upper bound on $|A|$. We use the inclusion-exclusion formula up to size 2 to derive the lower bound

$$\left|\bigcup_{p\in P} B_p^{a_p}\right| \geq \sum_{p\in P} \left|B_p^{a_p}\right| - \sum_{q,p\in P, q<p} \left|B_q^{a_q} \cap B_p^{a_p}\right|$$

Rewriting the RHS we get

$$\sum_{p\in P} \left[|B_p^{a_p}| - \sum_{q\in P, q<p} |B_p^{a_p} \cap B_q^{a_q}| \right]$$

Lower bounding this sum for any subset of primes $P_0 \subset P$ gives a lower bound for $|\bigcup_{p\in P_0} B_p^{a_p}|$ which in turn gives a lower bound for $|\bigcup_{p\in P} B_p^{a_p}|$. We define P_0 iteratively by the following algorithm:

1. $P_0 \leftarrow \emptyset$
2. $LB \leftarrow 0$
3. for $p \in P$ (ordered, up to d)
   (a) $LB_p \leftarrow \min_{a'_p \in \{0,1,\ldots,p-1\}} \left\{ |B_p^{a'_p}| - \sum_{q\in P_0} \max_{a'_q \in \{0,\ldots,q-1\}} \{|B_q^{a'_q} \cap B_p^{a'_p}|\} \right\}$
   (b) if $LB_p > 0$
      i. $P_0 \leftarrow P_0 \cup \{p\}$
      ii. $LB \leftarrow LB + LB_p$

The same algorithm can be applied when the initial set I is a subset of $[x, x + d - 1]$. Thus, we will do exhaustive search over the values of a_p for small primes p (say up to 17). Under any choice for residue for small p, we will calculate I as the subset that avoids such residue, and will estimate the lower bound according to the above algorithm. This gives an upper bound on the set of the admissible set assuming we have the correct value for the small primes residue. Taking a maximum over all possibilities gives an upper bound for the problem.

3 Results

The next table has upper bounds achieved by the above algorithm with two key parameters: exhaustive search prime bound, and the maximal prime in P_0 allowed.

d - Interval Length	Exhaustive Search	$\max\{p \in P_0\}$	Upper Bound on M_d
10000	13	75	1466
13900	17	100	1961
21500	17	100	2928
29500	17	100	3950
38000	17	100	4987
85870	13	150	10715
193000	13	200	22885

Using Claims 1.3 and 1.4 we derive the following lower bounds on h_k:

k_0	Engelsma Lower Bound	Lower Bound
672	4574	-
1000	6802	-
2000	13620	14082
3000	20434	21884
4000	27248	29746
5000	34068	38048
10719	73094	85878
22949	156614	193330