Shrinkage of De Morgan Formulae using Spectral Techniques

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\[ \text{Fix } x_1 = 0 \]
No coincidence!
A De Morgan formula, $F$, is a binary tree whose leaves are labeled with $\{x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n\}$ and whose internal nodes are labeled with AND, OR gates.

The size of the formula, $L(F)$, is the number of leaves in the tree.
A **De Morgan formula**, $F$, is a binary tree whose leaves are labeled with $\{x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n\}$ and whose internal nodes are labeled with AND, OR gates.

The **size** of the formula, $L(F)$, is the number of leaves in the tree.

for a Boolean **function** $f : \{0, 1\}^n \rightarrow \{0, 1\}$:

$$L(f) = \min_{F : F \text{ computes } f} L(F).$$
A De Morgan formula, $F$, is a binary tree whose leaves are labeled with $\{x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n\}$ and whose internal nodes are labeled with AND, OR gates.

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for a Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$:

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The Challenge: Find an explicit function $f$ (in P) with $L(f) = n^{\omega(1)}$.

Still open, even for NEXP.
A restriction is a function \( \rho : \{1, \ldots, n\} \rightarrow \{0, 1, *\} \).

Given a function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) the restricted function \( f|_{\rho} : \{0, 1\}^n \rightarrow \{0, 1\} \) is defined as:

\[
f|_{\rho}(x) = f(y) \ ; \ y_i = \begin{cases} x_i, & \rho(i) = * \\ \rho(i), & \text{otherwise} \end{cases}
\]
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$$f|_\rho(x) = f(y) \ ; \ y_i = \begin{cases} x_i, & \rho(i) = * \\ \rho(i), & \text{otherwise} \end{cases}.$$ 

$\mathcal{R}_p$ is a distribution over restrictions. For each $i \in [n]$, ind.

$$\rho(i) = \begin{cases} *, & \text{w.p. } p \\ 0, & \text{w.p. } (1-p)/2 \\ 1, & \text{w.p. } (1-p)/2 \end{cases}.$$

Theorem [Subbotovskaya '61]:

$$E_{\rho \sim \mathcal{R}_p}[L(f|_\rho)] = O(p^{1.5}L(f) + 1).$$
A **restriction** is a function $\rho : \{1, \ldots, n\} \rightarrow \{0, 1, *\}$.

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**Theorem [Subbotovskaya ’61]**:

$$\mathbf{E}_{\rho \sim \mathcal{R}_p} [L(f|_{\rho})] = O(p^{1.5} L(f) + 1).$$
Theorem [Sub ’61]: $E_{\rho \sim \mathcal{R}_p}[L(f|\rho)] = O(p^{1.5}L(f) + 1)$.

$\iff L(f) \geq \frac{\Omega(E[L(f|\rho)]) - 1}{p^{1.5}}$

$\implies L(\text{Parity}) \geq \Omega(n^{1.5})$. 

[Andreev ’87] gave an explicit function $A: \{0, 1\}^n \rightarrow \{0, 1\}$ such that for $p \approx \log(n)/n$

$E_{\rho \sim \mathcal{R}_p}[L(A|\rho)] \geq \Omega(n\log\log n)$.

$\implies L(A) \geq \Omega(n^{2.5}\text{polylog}(n))$.

[H˚ astad ’93] improved Subbotovskaya’s result:

$E_{\rho \sim \mathcal{R}_p}[L(f|\rho)] = O(p^{2}\left(1 + \log\frac{3}{2}(1/p)\right)L(f) + 1)$

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Theorem [Sub '61]: $E_{\rho \sim \mathcal{R}_p}[L(f|\rho)] = O(p^{1.5}L(f) + 1)$.

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Theorem [Sub ’61]: $E_{\rho \sim \mathcal{R}_p}[L(f|\rho)] = O(p^{1.5}L(f) + 1)$.

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[Håstad ’93] improved Subbotovskaya’s result:

$$E_{\rho \sim \mathcal{R}_p}[L(f|\rho)] = O \left( p^2 \left( 1 + \log^{3/2}(1/p) \right) L(f) + 1 \right)$$
Andreev’s Function

A : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}

x_1 x_2 x_3 x_4 x_5 \ldots x_n

z_1 = \bigoplus_{j=1}^{m} y_{1j}

z_2 = \bigoplus_{j=1}^{m} y_{2j}

\ldots

z_r = \bigoplus_{j=1}^{m} y_{rj}

y_{11} y_{12} y_{13} \ldots y_{1m}

y_{21} y_{22} y_{23} \ldots y_{2m}

\ldots

y_{r1} y_{r2} y_{r3} \ldots y_{rm}

r = \log n

m = \frac{n}{r}

\text{output } x(z_1, \ldots, z_r)
Andreev’s Function

Let $A_h(y) = A(tt(h), y)$ for the hardest $h : \{0, 1\}^{\log n} \rightarrow \{0, 1\}$, then $L(A) \geq L(A_h)$. 
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For $p = \frac{\log \log(n) \log(n)}{n}$, \( E_{\rho \sim \mathcal{R}_p} [L(A_h | \rho)] \geq \Omega(L(h)) \geq \Omega \left( \frac{n}{\log \log(n)} \right) \)
Main Theorem:

$$\mathbb{E}_{\rho \sim \mathcal{R}_p} [L(f|\rho)] = O(p^2L(f) + 1).$$
Lower Bounds from Shrinkage
Our Results

Main Theorem:

$$\mathbb{E}_{\rho \sim \mathcal{R}_p} [L(f|\rho)] = O(p^2L(f) + 1).$$

The result is \textbf{tight} as demonstrated by \textit{Parity}. 
**Main Theorem:**

\[
\mathbb{E}_{\rho \sim \mathcal{R}_p} [L(f|\rho)] = O(p^2 L(f) + 1).
\]

The result is **tight** as demonstrated by *Parity*.

**Applications:**

- Theorem \( \implies \) \( L(A) \geq \Omega \left( \frac{n^3}{\log^2(n) \log^3 \log(n)} \right) \).
- This is **tight** up to \( \log^3 \log(n) \) factor.
- Theorem \( \implies \) improved analysis of **average case** lower bound by [Komargodski Raz T ’13].
Every function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ has a unique (Fourier) representation as a multilinear polynomial

$$f(x) = \sum_{S \subseteq [n]} \hat{f}(S) \prod_{i \in S} x_i$$

where $\hat{f}(S) = \mathbb{E}_x[f(x) \prod_{i \in S} x_i]$. 

**Parseval's identity:**

$$1 = \mathbb{E}_x[f(x)^2] = \sum_{S \subseteq [n]} \hat{f}(S)^2.$$ 

**Fourier weights:**

$$W_k[f] \equiv \sum_{S : |S| = k} \hat{f}(S)^2.$$ 

**Degree:**

$$\deg(f) \equiv \max \{|S| : \hat{f}(S) \neq 0\}.$$ 

**Granularity:**

if $\deg(f) = d$ then $\forall S : \hat{f}(S)$ is an integer·$2^{-d}$.

**Approximate Degree:**

$$\tilde{\deg}_\epsilon(f) \equiv \min \{\deg(p) : \forall x : |p(x) - f(x)| \leq \epsilon\}.$$
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Preliminaries

Discrete Fourier Transform and Approximate Degree

• Every function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ has a unique (Fourier) representation as a multilinear polynomial

$$f(x) = \sum_{S \subseteq [n]} \hat{f}(S) \prod_{i \in S} x_i$$

where $\hat{f}(S) = E_x[f(x) \prod_{i \in S} x_i]$.

• Parseval’s identity: $1 = E_x[f(x)^2] = \sum_S \hat{f}(S)^2$.

• Fourier weights: $W^k[f] \triangleq \sum_{S : |S| = k} \hat{f}(S)^2$. 
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- **Degree:** $\deg(f) \triangleq \max\{|S| : \hat{f}(S) \neq 0\}$.
- **Granularity:** if $\deg(f) = d$ then $\forall S : \hat{f}(S) = \text{integer} \cdot 2^{-d}$. 
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- **Parseval’s identity:** $1 = \mathbb{E}_x[f(x)^2] = \sum_S \hat{f}(S)^2$.
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- **Degree of $f$:** $\deg(f) \triangleq \max\{|S| : \hat{f}(S) \neq 0\}$.
- **Granularity:** if $\deg(f) = d$ then $\forall S : \hat{f}(S) = \text{integer} \cdot 2^{-d}$.
- **Approximate Degree:** $\tilde{\deg}_\epsilon(f) \triangleq \min \{\deg(p) \mid \forall x : |p(x) - f(x)| \leq \epsilon\}$. 
\[ \overline{\deg_{1/3}(f)} \leq O(\sqrt{L(f)}) \]
Proof Overview

1. \( \widetilde{\deg}_{1/3}(f) \leq O(\sqrt{L(f)}) \)

2. Exponentially small Fourier tails: \( W^k[f] \leq e^{-k/\sqrt{L(f)}} \).
   (where \( W^k[f] \triangleq \sum_{|S| > k} \hat{f}(S)^2 \))
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3. Degree shrinkage: $\Pr_{\rho \sim \mathcal{R}_p}[^{\deg}(f | \rho) = d] \leq \left(4p\sqrt{L(f)}\right)^d$. 

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4. The low probability case:
   if $p \leq \frac{1}{1000\sqrt{L(f)}}$, then $E_{\rho \sim \mathcal{R}_p}[L(f|\rho)] = O(1)$.
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1. \( \overline{\deg_{1/3}}(f) \leq O(\sqrt{L(f)}) \)

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4. The low probability case:
   if \( p \leq \frac{1}{1000\sqrt{L(f)}} \), then \( \mathbb{E}_{\rho \sim \mathcal{R}_p}[L(f|\rho)] = O(1) \).

5. The general case: \( \mathbb{E}_{\rho \sim \mathcal{R}_p}[L(f|\rho)] = O(p^2L(f) + 1) \).
Step 1
Approximate Degree is at most square root of Formula size

- **Recall:** $\tilde{\deg}_\epsilon(f) \triangleq \min \{ \deg(p) \mid \forall x : |p(x) - f(x)| \leq \epsilon \}$
- **The Black Box model**

```
i → x ∈ \{0,1\}^n
```

$\tilde{\deg}_{1/3}(f) \leq O(Q_2(f))$

([Beals Buhrman Cleve Mosca de Wolf '98])

$= \Theta(\text{ADV}^\pm(f))$  \hspace{1cm} ([Reichardt '11])

$\leq O(\sqrt{L(f)})$  \hspace{1cm} ([Høyer Lee Spalek '07])

- **Open:** is there a classical proof for $\tilde{\deg}_{1/3}(f) \leq O(\sqrt{L(f)})$?
Claim [Buhrman Newman Röhrig de Wolf ’05]:

$$\forall t : \widetilde{\deg}_{e^{-t}}(f) \leq \widetilde{\deg}_{1/3}(f) \cdot t.$$ 

Proof Idea: If \( p(x) \) approx. \( f(x) \) up to err. 1/3 then \( \text{MAJ}_t(p(x), \ldots, p(x)) \) approx. \( f(x) \) up to err. \( \exp(-t) \).
Step 2
Exponentially small Fourier tails

Claim [Buhrman Newman Röhrig de Wolf ’05]:
\[ \forall t : \widetilde{\deg}_{e^{-t}}(f) \leq \widetilde{\deg}_{1/3}(f) \cdot t. \]

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Claim:
\[ \forall k : W^k[f] \leq e^{-k/\widetilde{\deg}_{1/3}(f)} \]

Proof Idea:
1. An \( L_\infty \) approx. is in particular an \( L_2 \) approx.
2. \( f^{\leq k}(x) \triangleq \sum_{|S| \leq k} \hat{f}(S) \prod_{i \in S} x_i \)
   is the best \( L_2 \) approx. among degree \( \leq k \) polynomials.
Step 3
Exponentially small Fourier tails implies “Degree Shrinkage”

**Theorem**

Let $t$ be some parameter. If $\forall k : W^k[f] \leq e^{-k/t}$ then

$$\forall p, d : \Pr_{\rho \sim R_p}[\deg(f|_\rho) = d] \leq (4pt)^d$$
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Proof Idea:

1. Under restriction the Fourier mass at level $d$ is extremely small: $E_{\rho \sim R_p}[W^d[f|\rho]] \leq (pt)^d$. 
### Theorem

Let $t$ be some parameter. If $\forall k : W^k[f] \leq e^{-k/t}$ then

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### Proof Idea:

1. Under restriction the Fourier mass at level $d$ is extremely small: $E_{\rho \sim R_p}[W^d[f|\rho]] \leq (pt)^d$.

2. If $f|\rho$ is of degree $d$ then Fourier mass at level $d$ is at least $4^{-d}$ (Granularity).
Step 3
Exponentially small Fourier tails implies “Degree Shrinkage”

Theorem
Let \( t \) be some parameter. If \( \forall k : W^{>k}[f] \leq e^{-k/t} \) then

\[
\forall p, d : \Pr_{\rho \sim R_p} [\deg(f|_\rho) = d] \leq (4pt)^d
\]

Proof Idea:
1. Under restriction the Fourier mass at level \( d \) is extremely small: \( \mathbf{E}_{\rho \sim R_p} [W^d[f|_\rho]] \leq (pt)^d \).
2. If \( f|_\rho \) is of degree \( d \) then Fourier mass at level \( d \) is at least \( 4^{-d} \) (Granularity).
3. \( \implies \) the probability that the degree is \( d \) must be small:

\[
(pt)^d \geq \mathbf{E}_{\rho \sim R_p} [W^d[f|_\rho]]
\]
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Exponentially small Fourier tails implies “Degree Shrinkage”

**Theorem**

Let $t$ be some parameter. If $\forall k : \mathbf{W}^{>k}[f] \leq e^{-k/t}$ then

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**Proof Idea:**

1. Under restriction the Fourier mass at level $d$ is extremely small: $\mathbf{E}_{\rho \sim R_p} [\mathbf{W}^d[f|\rho]] \leq (pt)^d$.
2. If $f|\rho$ is of degree $d$ then Fourier mass at level $d$ is at least $4^{-d}$ (Granularity).
3. $\implies$ the probability that the degree is $d$ must be small:

$$\mathbf{E}_{\rho \sim R_p} [\mathbf{W}^d[f|\rho]] \leq (pt)^d \cdot \Pr_{\rho \sim R_p} [\deg(f|\rho) = d]$$
Step 3.1
Exponentially small Fourier tails implies “Degree Shrinkage”

Claim [Linial Mansour Nisan ’89]

\[ E_{\rho \sim \mathcal{R}_p}[W^d[f|\rho]] = \sum_{k \geq d} W^k[f] \cdot \Pr[\text{Bin}(k, p) = d] \]

\[ W^k[f] \quad \text{and} \quad E[W^d[f|\rho]] \]
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Exponentially small Fourier tails implies “Degree Shrinkage”

Claim [Linial Mansour Nisan ’89]

$$E_{\rho \sim \mathcal{R}_p}[W^d[f | \rho]] = \sum_{k \geq d} W^k[f] \cdot \Pr[\text{Bin}(k, p) = d]$$

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\[ W^k[f] \quad \text{and} \quad E[W^d[f|\rho]] \]

\[ \forall k : W^{>k}[f] \leq e^{-k/t} \implies \forall d : E_{\rho \sim \mathcal{R}_p}[W^d[f|\rho]] \leq (pt)^d. \]
Step 4 - The low probability case

- **Recall:** $\Pr[\deg(f | \rho) = d] \leq (4p\sqrt{L(f)})^d$.
- **Fact:** If $\deg(g) = d$ then $L(g) \leq 32^d$.

---

**Theorem**

If $p \leq \frac{1}{1000\sqrt{L(f)}}$ then $\mathbb{E}[L(f | \rho)] = O(1)$.
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**Theorem**

If \( p \leq \frac{1}{1000\sqrt{L(f)}} \) then \( \mathbb{E}[L(f|\rho)] = O(1) \).

**Proof:**

\[
\mathbb{E}[L(f|\rho)] = \sum_{d=0}^{n} \Pr[\deg(f|\rho) = d] \cdot \mathbb{E}[L(f|\rho)|\deg(f|\rho) = d]
\leq \sum_{d=0}^{n} \left( \frac{1}{250} \right)^d \cdot 32^d = O(1)
\]
Step 4 - The low probability case

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- **Fact:** If \( \deg(g) = d \) then \( L(g) \leq 32^d \).

**Theorem**

If \( p \leq \frac{1}{1000\sqrt{L(f)}} \) then \( \mathbb{E}[L(f|\rho)] = O(1) \).

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\mathbb{E}[L(f|\rho)] = \sum_{d=0}^{n} \Pr[\deg(f|\rho) = d] \cdot \mathbb{E}[L(f|\rho)|\deg(f|\rho) = d]
\leq \sum_{d=0}^{n} \left( \frac{1}{250} \right)^d \cdot 32^d = O(1)
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Step 5 - The general case

- Based on [Impagliazzo Meka Zuckerman ’12]: For a given $\ell$, we can transform any formula $F$ into a nicer formula $F'$,

$$m = O\left(\frac{L(F)}{\ell} + 1\right), \text{ and } \forall i : L(F_i) \leq \ell.$$
Step 5 - The general case

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$$m = O\left(\frac{L(F)}{\ell} + 1\right), \text{ and } \forall i : L(F_i) \leq \ell.$$

- Pick $\ell := \frac{1}{1000^2 p^2}$. In each subformula $E[L(F_i|\rho)] = O(1)$.

$$E[L(F'|\rho)] \leq \sum_i E[L(F_i|\rho)] = O(m) = O(p^2 L(F) + 1).$$
Step 5 - The general case

- Based on [Impagliazzo Meka Zuckerman '12]: For a given \( \ell \), we can transform any formula \( F \) into a nicer formula \( F' \),

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m = O \left( \frac{L(F)}{\ell} + 1 \right), \quad \text{and} \quad \forall i : L(F_i) \leq \ell.
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- Pick \( \ell := \frac{1}{1000^2 p^2} \). In each subformula \( E[L(F_i|\rho)] = O(1) \).

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E[L(F'|\rho)] \leq \sum_i E[L(F_i|\rho)] = O(m) = O(p^2 L(F) + 1).
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Recap

- Main motivation: $\text{P vs. NC}^1$.
- $O(p^2)$ shrinkage implies $\frac{n^3}{(\log n)^{2+o(1)}}$ lower bounds for $A$. 

Open Question: Breaking the $n^3$ barrier. 

Open Question [H˚astad, Paterson-Zwick]: What is the shrinkage exponent of monotone De Morgan formulae?

Enough to prove the low probability case.
Main motivation: \( P \) vs. \( NC^1 \).

\( \mathcal{O}(p^2) \) shrinkage implies \( \frac{n^3}{(\log n)^{2+o(1)}} \) lower bounds for \( A \).

New proof for \( \mathcal{O}(p^2) \) shrinkage:

1. \( \widetilde{\deg}(f) \leq O(\sqrt{L(f)}) \) using quantum query complexity
2. Exponentially small Fourier tails
3. Degree shrinkage
4. The low probability case
5. The general case
Recap

Main motivation: $\mathbf{P}$ vs. $\mathbf{NC}^1$.

$O(p^2)$ shrinkage implies $\frac{n^3}{(\log n)^{2+o(1)}}$ lower bounds for $A$.

New proof for $O(p^2)$ shrinkage:

1. $\widetilde{\deg}(f) \leq O(\sqrt{L(f)})$ using quantum query complexity
2. Exponentially small Fourier tails
3. Degree shrinkage
4. The low probability case
5. The general case

Open Question: Breaking the $n^3$ barrier.
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Open Question [Håstad, Paterson-Zwick]: What is the shrinkage exponent of monotone De Morgan formulae?

Enough to prove the low probability case.
Thank You