

Social Choice – Influences, Effects, Noise Stability and Arrow’s Theorem

1 Boolean Functions as Voting Rules

In this chapter, we look at Boolean functions from the point of view of social choice. We view a Boolean function

$$f : \{-1, 1\}^n \rightarrow \{-1, 1\}$$

as a voting rule mapping n individual votes into a collective decision. We begin by giving some examples of possible voting rules:

1. Majority vote - for n odd, the Majority-vote function is defined by $\text{MAJ}_n(x_1, \dots, x_n) = \text{sgn}(x_1 + \dots + x_n)$. Note that since n is odd the sum is never 0. ¹
2. The AND function² $\text{AND}_n(x_1, \dots, x_n) = \begin{cases} -1, & \text{if } x = (-1, \dots, -1) \\ +1, & \text{otherwise.} \end{cases}$
3. The OR function $\text{OR}_n(x_1, \dots, x_n) = \begin{cases} +1, & \text{if } x = (+1, \dots, +1) \\ -1, & \text{otherwise.} \end{cases}$
4. The Dictator function $\chi_i(x_1, \dots, x_n) = x_i$.

Definition 1 (*k-junta*). A function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ is a *k-junta* if f depends on at most k of its input variables. That is, there exists indices $1 \leq i_1 < i_2 < \dots < i_k \leq n$ and a function $g : \{-1, 1\}^k \rightarrow \{-1, 1\}$ such that

$$f(x_1, \dots, x_n) = g(x_{i_1}, \dots, x_{i_k}).$$

Definition 2 (*LTF*). A function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ is a *linear-threshold-function* (*LTF*, *weighted-majority*) if there exists real numbers a_0, \dots, a_n such that

$$f(x_1, \dots, x_n) = \text{sgn}(a_0 + a_1x_1 + a_2x_2 + \dots + a_nx_n).$$

Observe that the Majority, AND, OR, and the Dictator functions are all LTFs.

5. The Tribes function with size s and width w . We have s groups (or tribes), each consists of distinct w bits. We take the AND of each tribe and then take the OR of the s results. This gives a function on $n = ws$ bits, $\text{Tribes}_{w,s} : \{-1, 1\}^n \rightarrow \{-1, 1\}$ defined by

$$\text{Tribes}_{w,s}(x) = \text{OR}(\text{AND}(x_1, \dots, x_w), \text{AND}(x_{w+1}, \dots, x_{2w}), \dots, \text{AND}(x_{w(s-1)+1}, \dots, x_{ws})).$$

¹We can extend the Majority function to even n , in which case we will decide arbitrarily in case of a tie, for example we will always output 1 in case of a tie (i.e., when $x_1 + \dots + x_n = 0$).

²Recall that we represent **TRUE** by -1 , and **FALSE** by 1 .

A typical choice of parameters will be $s \approx 2^w \ln(2)$, so that

$$\Pr_{\mathbf{x} \sim \{-1,1\}^n} [\text{Tribes}_{w,s}(\mathbf{x}) = 1] \approx 1/2.$$

Recall the simulation we had in class. Each of us needed to decide whether they prefer Vanilla or Chocolate as an ice-cream flavor. We first computed the Tribes function by mapping Vanilla to **FALSE** = +1 and Chocolate to **TRUE** = -1. The result was that no tribe was unanimous on selecting Chocolate, hence the collective decision was Vanilla. Then, we switched roles between Vanilla and Chocolate, mapping Vanilla to **TRUE** = -1 and Chocolate to **FALSE** = +1, and then realized that there is a tribe that was unanimous on selecting Vanilla. Indeed, also in this case the collective decision was Vanilla. (In general, it could have been that the two decisions would be different. For example, if no tribe was unanimous. This means that Tribes is not an odd function – see the definition below).

Desirable Properties of a Voting Scheme

- **Not a Junta:** for all $i \in [n]$ there exists an input x such that flipping the i -th coordinate flips the value of f . Namely, $f(x) \neq f(x^{\oplus i})$, where

$$x^{\oplus i} = (x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_n).$$

- **Symmetric:** for all permutations $\pi \in S_n$ (i.e., permutations $\pi : [n] \rightarrow [n]$), we have $f(x) = f(x^\pi)$ for all x , where

$$x^\pi = (x_{\pi(1)}, \dots, x_{\pi(n)}).$$

Observe that a symmetric function depends only on $\sum_{i=1}^n x_i$, i.e., the number of 1's in the input.

- **Monotone:** for all $x, y \in \{-1, 1\}^n$ such that $x \leq y$ we have $f(x) \leq f(y)$ (where by $x \leq y$ we mean that all the coordinates satisfy $x_i \leq y_i$). Intuitively, when flipping an input bit from -1 to +1, the value of f can stay the same or be flipped in the positive direction.³
- **Unanimous:** $f(1, 1, \dots, 1) = 1$ and $f(-1, -1, \dots, -1) = -1$.
- **Odd:** $f(-x) = -f(x)$. Observe that in an odd function $\Pr_{\mathbf{x} \sim \{-1,1\}^n} [f(\mathbf{x}) = 1] = 1/2$.

We checked which properties are satisfied by Majority, AND, OR, Dictator and Tribes. Majority satisfied all the desirable properties, and in fact it is the only Boolean function that is symmetric, monotone, unanimous and odd. (Exercise: Verify it! This is called May's Theorem).

Next, we relax the notion of a symmetric function.

Definition 3 (Symmetry Group). *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$. The symmetry group of f , or automorphism group of f , is the following group⁴*

$$\text{Aut}(f) = \{\pi \in S_n : \forall x \in \{-1, 1\}^n : f(x) = f(x^\pi)\}.$$

Definition 4 (Symmetric Function, Transitive-Symmetric Function). *We say that f is symmetric if $\text{Aut}(f) = S_n$. We say that f is transitive-symmetric if for all $i, j \in [n]$ there exists a permutation $\pi \in \text{Aut}(f)$ such that $\pi(i) = j$.*

Observe that the Tribes function is transitive-symmetric, but not symmetric.

³Elizabeth mentioned Truthfulness – that all voters are incentivized to reveal their true preference. Indeed, as observed by Orr, in a monotone voting rule, truthfulness is achieved.

⁴Exercise: verify that this is indeed a subgroup of S_n .

2 Influences, Effects, Derivatives

Definition 5 (Pivotal Coordinate). *We say that coordinate $i \in [n]$ is pivotal (or sensitive) for $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ on input x if $f(x) \neq f(x^{\oplus i})$. (Recall that $x \oplus i = (x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_n)$).*

Definition 6 (Influence). *The influence of coordinate $i \in [n]$ on $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ is defined as the probability that i is pivotal on a uniformly random input \mathbf{x} . Namely,*

$$\mathbf{Inf}_i[f] = \Pr_{\mathbf{x} \sim \{-1, 1\}^n} [f(\mathbf{x}) \neq f(\mathbf{x}^{\oplus i})].$$

Fact 7. *For $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$:*

$$\mathbf{Inf}_i[f] = \frac{\# \text{ of pivotal edges for } f \text{ in direction } i}{\# \text{ of edges in direction } i} = \frac{\# \text{ of pivotal edges for } f \text{ in direction } i}{2^{n-1}}.$$

Let's see a few examples.

1. The dictator function χ_i has $\mathbf{Inf}_i[\chi_i] = 1$ and $\mathbf{Inf}_j[\chi_i] = 0$ for any $i \neq j$.
2. Majority of 3 bits, MAJ₃, has in each direction 2 pivotal edges, hence

$$\mathbf{Inf}_1[\text{MAJ}_3] = \mathbf{Inf}_2[\text{MAJ}_3] = \mathbf{Inf}_3[\text{MAJ}_3] = \frac{2}{4} = \frac{1}{2}$$

3. More generally, if n is odd, then all influences of Majority are the same since the function is symmetric, and they are equal to

$$\mathbf{Inf}_1[\text{MAJ}_n] = \Pr_{\mathbf{x} \sim \{-1, 1\}^n} [\mathbf{x}_2 + \mathbf{x}_3 + \dots + \mathbf{x}_n = 0] = \frac{\binom{n-1}{(n-1)/2}}{2^{n-1}} \approx \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}} \approx 0.8 \cdot \frac{1}{\sqrt{n}}$$

4. What about the influence in the Tribes function? Also, here all the influences are the same (see the next claim) and they are equal to

$$\begin{aligned} \mathbf{Inf}_1[\text{Tribes}_{w,s}] &= \Pr[\text{all other members in } x_1 \text{'s tribe are TRUE}] \\ &\quad \cdot \Pr[\text{all other tribes not all TRUE}] \\ &= 2^{-(w-1)} \cdot (1 - 2^{-w})^{s-1} \end{aligned}$$

If we set $s = \lceil 2^w \ln(2) \rceil$ so that

$$\Pr[\text{Tribes}_{w,s}(\mathbf{x}) = 1] = (1 - 2^{-w})^s \approx \frac{1}{2},$$

then we get $\mathbf{Inf}_i[\text{Tribes}_{w,s}] = 2^{-(w-1)} \cdot (1 - 2^{-w})^{s-1} \approx 2^{-(w-1)} \cdot \frac{1}{2} = 2^{-w} \approx w \ln(2)/n$. Denoting by $n = sw = w \cdot \lceil 2^w \cdot \ln(2) \rceil$ we see that $w \ln(2)/n \approx \ln(n)/n$. In fact, later we will see that this is the lowest influences one can get in a voting rule that is almost balanced, i.e., one where $\Pr[f(\mathbf{x}) = 1]$ is bounded away from 0 and 1.

Theorem 8 (Kahn-Kalai-Linial'88). *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}^n$. Then, there exists a coordinate $i \in [n]$ such that $\mathbf{Inf}_i[f] \geq \Omega\left(\frac{\ln(n)}{n}\right) \cdot \mathbf{Var}[f]$.⁵ In particular, if $\Pr[f(\mathbf{x}) = 1] \in [0.1, 0.9]$, then $\mathbf{Inf}_i[f] \geq \Omega\left(\frac{\ln(n)}{n}\right)$.*

⁵Recall that $\mathbf{Var}[f] = \mathbf{E}[f(\mathbf{x})^2] - \mathbf{E}[f(\mathbf{x})]^2$ and for Boolean f it equals $4 \cdot \alpha \cdot (1 - \alpha)$ where $\alpha = \Pr[f(\mathbf{x}) = 1]$.

The fact that all the influences of Tribes are equal is a special case of the following lemma.

Lemma 9 (In a transitive-symmetric function, all influences are equal). *If $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ is transitive-symmetric, then for all $i, j \in [n]$ we have $\mathbf{Inf}_i[f] = \mathbf{Inf}_j[f]$.*

Proof. Fix $i, j \in [n]$. Since f is transitive-symmetric there exists a permutation $\pi \in \text{Aut}(f)$ that maps j to i .

$$\mathbf{Inf}_i[f] = \Pr_{\mathbf{x} \sim \{-1, 1\}^n} [f(\mathbf{x}) \neq f(\mathbf{x}^{\oplus i})] = \Pr[f(\mathbf{x}^\pi) \neq f((\mathbf{x}^{\oplus i})^\pi)]$$

Observe that $(x^{\oplus i})^\pi = (x^\pi)^{\oplus j}$, thus

$$\mathbf{Inf}_i[f] = \Pr[f(\mathbf{x}^\pi) \neq f((\mathbf{x}^\pi)^{\oplus j})] = \mathbf{Inf}_j[f]$$

since \mathbf{x}^π is distributed uniformly at random over $\{-1, 1\}^n$. \square

The next definition is *non-standard* (for example, does not appear in O’Donnell’s book) but captures an interesting variant of influence.

Definition 10 (Effect). *The effect of coordinate $i \in [n]$ on $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ is defined to be*

$$\mathbf{Eff}_i[f] = \Pr_{\mathbf{x} \sim \{-1, 1\}^n} [f(\mathbf{x}) = 1 | \mathbf{x}_i = 1] - \Pr_{\mathbf{x} \sim \{-1, 1\}^n} [f(\mathbf{x}) = 1 | \mathbf{x}_i = -1].$$

We observe that the effect measures how much voter i can change the probability of outputting 1 assuming all other voters pick their votes uniformly at random. To compare between influence and effect, it is good to think of the following intuitive explanation: effect measure how much voter i can change the outcome if she casts her vote **first** and then all others decide randomly, while influence measures how much voter i can change the outcome if she casts her vote **last** after seeing all other votes. Due to this intuitive explanation, it is clear that the influence is always bigger than the effect, since we are making the decision with more knowledge. Note however, that for monotone functions the two measures are equal. (We will see a different proof for this later).

We derive a nice formula for the effect in terms of the Fourier coefficients of f .

Fact 11. *For $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ we have $\mathbf{Eff}_i[f] = \widehat{f}(\{i\})$.*

Proof.

$$\begin{aligned} \widehat{f}(\{i\}) &= \mathbf{E}_{\mathbf{x}} [f(\mathbf{x}) \cdot \mathbf{x}_i] && \text{(Inversion Formula)} \\ &= \frac{1}{2} \cdot \mathbf{E}_{\mathbf{x}} [f(\mathbf{x}) | \mathbf{x}_i = 1] - \frac{1}{2} \cdot \mathbf{E}_{\mathbf{x}} [f(\mathbf{x}) | \mathbf{x}_i = -1] && \text{(Total Expectation Rule)} \\ &= \frac{1}{2} \cdot \mathbf{E}_{\mathbf{x}} [(2 \cdot \mathbf{1}_{\{f(\mathbf{x})=1\}} - 1) | \mathbf{x}_i = 1] - \frac{1}{2} \cdot \mathbf{E}_{\mathbf{x}} [(2 \cdot \mathbf{1}_{\{f(\mathbf{x})=1\}} - 1) | \mathbf{x}_i = -1] \\ &= (\Pr[f(\mathbf{x}) = 1 | \mathbf{x}_i = 1] - \frac{1}{2}) - (\Pr[f(\mathbf{x}) = 1 | \mathbf{x}_i = -1] - \frac{1}{2}) \\ &= \Pr[f(\mathbf{x}) = 1 | \mathbf{x}_i = 1] - \Pr[f(\mathbf{x}) = 1 | \mathbf{x}_i = -1]. && \square \end{aligned}$$

Note that effects and influences can be different for non-monotone functions. For example, for the Parity function $\chi_{[n]}(x) = \prod_{i=1}^n x_i$, all effects are 0 and all influences are 1. Effect captures more naturally how much a single voter “effects” the outcome of an election, since it is a reasonable assumption that voter i do not know what others are voting before casting her vote. Effect is equal to influence for monotone functions, which are the “reasonable” voting rules.

Next, we wish to derive a nice formula for the Influence in terms of the Fourier coefficients of the function. For that, we need to define the discrete derivative.

2.1 Derivatives and Fourier Formula for Influences

Definition 12 (Derivative). *The derivative operator D_i maps a function $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ to the function $D_i f : \{-1, 1\}^n \rightarrow \mathbb{R}$ defined by*

$$D_i f(x) = \frac{f(x^{(i \rightarrow 1)}) - f(x^{(i \rightarrow -1)})}{2},$$

where we denote by $x^{(i \rightarrow 1)} = (x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$ for $b \in \{-1, 1\}$.

Note that if f is Boolean-valued, i.e., $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$, then

$$D_i f(x) = \begin{cases} \pm 1, & \text{if coordinate } i \text{ is pivotal on input } x \\ 0, & \text{otherwise} \end{cases}.$$

Thus, $(D_i f(x))^2 = \mathbf{1}_{\{\text{coordinate } i \text{ is pivotal on input } x\}}$ and we thus get

$$\mathbf{Inf}_i[f] = \mathbf{E}_{\mathbf{x} \sim \{-1, 1\}^n} [(D_i f(x))^2] \quad (1)$$

In fact, for non-Boolean functions $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ we take Eq. (1) as the definition of influence.

Fact 13. *Let $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ with Fourier expansion $f(x) = \sum_{S \subseteq [n]} \hat{f}(S) \cdot \chi_S(x)$.⁶ Then,*

$$D_i f(x) = \sum_{S \ni i} \hat{f}(S) \chi_{S \setminus \{i\}}(x).$$

Note that this agrees with the standard notion of the partial derivative of a multilinear polynomial with respect to variable x_i .

Proof. First, observe that f is a linear operator on functions mapping $\{-1, 1\}^n$ to \mathbb{R} . That is, $D_i(f + g) = D_i(f) + D_i(g)$ for all $f, g : \{-1, 1\}^n \rightarrow \mathbb{R}$. Similarly, and $D_i(\alpha f) = \alpha D_i(f)$ for all functions $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ and scalars $\alpha \in \mathbb{R}$.

Thus, it suffices to prove that

$$D_i \chi_S(x) = \begin{cases} 0 & i \notin S \\ \chi_{S \setminus \{i\}}(x) & i \in S. \end{cases}$$

The first case is obvious since if $i \notin S$ then $\chi_S(x^{(i \rightarrow 1)})$ and $\chi_S(x^{(i \rightarrow -1)})$ are the same. The second case can be verified by inspection as

$$D_i \chi_S(x) = \frac{\chi_S(x^{(i \rightarrow 1)}) - \chi_S(x^{(i \rightarrow -1)})}{2} = \frac{1 \cdot \prod_{j \in S \setminus \{i\}} x_j - (-1) \cdot \prod_{j \in S \setminus \{i\}} x_j}{2} = \prod_{j \in S \setminus \{i\}} x_j. \quad \square$$

We are ready to state the Fourier formula for $\mathbf{Inf}_i[f]$.

Lemma 14. $\mathbf{Inf}_i[f] = \sum_{S \ni i} \hat{f}(S)^2$.

Proof. Apply Parseval's identity on the Fourier expansion of $D_i f$. □

We also can show that the effect is the expected derivative.

⁶Recall that $\chi_S(x) = \prod_{i \in S} x_i$.

Fact 15. $\mathbf{Eff}_i[f] = \widehat{f}(\{i\}) = \mathbf{E}_{\mathbf{x} \sim \{-1,1\}^n} [D_i f(\mathbf{x})]$.

Proof. By the inversion formula $\mathbf{E}_{\mathbf{x} \sim \{-1,1\}^n} [D_i f(\mathbf{x})] = \widehat{D_i f}(\emptyset) = \widehat{f}(\{i\})$ and we have already shown that the latter equal $\mathbf{Eff}_i[f]$. \square

Corollary 16. *For any function $f : \{-1,1\}^n \rightarrow \{-1,1\}$ we have $\mathbf{Eff}_i[f] \leq \mathbf{Inf}_i[f]$ and equality holds iff f is monotone in the i -th direction*

Proof.

$$\mathbf{Eff}_i[f] = \mathbf{E}_{\mathbf{x} \sim \{-1,1\}^n} [D_i f(\mathbf{x})] \leq \mathbf{E}_{\mathbf{x} \sim \{-1,1\}^n} [(D_i f(\mathbf{x}))^2] = \mathbf{Inf}_i[f]$$

where the inequality holds since $D_i f(x)$ of a Boolean function is either $-1, 0$ or 1 . Note that the equality holds if and only if all $D_i f(x) \geq 0$ which is equivalent to saying that the function is monotone in the i -th direction. \square

2.2 Total Influence, Total Effect

A natural measure of a Boolean function is the sum of all influences of its variables. Similarly, we can measure the sum of all the effects of its variables.

Definition 17. *The total-influence of a function $f : \{-1,1\}^n \rightarrow \mathbb{R}$ is defined as*

$$\mathbf{I}[f] = \sum_{i=1}^n \mathbf{Inf}_i[f].$$

The total-effect of a function $f : \{-1,1\}^n \rightarrow \mathbb{R}$ is defined as

$$\mathbf{Eff}[f] = \sum_{i=1}^n \mathbf{Eff}_i[f] = \sum_{i=1}^n \widehat{f}(\{i\}).$$

For example, the Majority function has total-influence (and total-effect) $\approx \sqrt{\frac{2}{\pi}} \cdot \sqrt{n}$. For Tribes the total-influence (and total-effect) is $\approx \ln(n)$. For Parity on n variables the total-influence is n , but the total-effect is 0 .

Next, we derive a few alternative interpretations of the total influence of a Boolean function.

Lemma 18. *Let $f : \{-1,1\}^n \rightarrow \{-1,1\}$. Then,*

$$\mathbf{I}[f] = \mathbf{E}_{\mathbf{x}} [\mathit{sens}_f(\mathbf{x})] \tag{2}$$

where $\mathit{sens}_f(x)$ equals the number of pivotal coordinates on x (this is called the sensitivity of f at x).

In words, Equation (2) states that the **total influence equals the average sensitivity**.

Proof. The proof is just a simple double-counting argument:

$$\begin{aligned} \mathbf{I}[f] &= \sum_{i=1}^n \mathbf{Inf}_i[f] = \sum_{i=1}^n \mathbf{Pr}_{\mathbf{x} \sim \{-1,1\}^n} [f(\mathbf{x}) \neq f(\mathbf{x}^{\oplus i})] = \sum_{i=1}^n \mathbf{E}_{\mathbf{x} \sim \{-1,1\}^n} [\mathbf{1}_{\{f(\mathbf{x}) \neq f(\mathbf{x}^{\oplus i})\}}] \\ &= \mathbf{E}_{\mathbf{x} \sim \{-1,1\}^n} \left[\sum_{i=1}^n \mathbf{1}_{\{f(\mathbf{x}) \neq f(\mathbf{x}^{\oplus i})\}} \right] = \mathbf{E}_{\mathbf{x} \sim \{-1,1\}^n} [\mathit{sens}_f(\mathbf{x})] \end{aligned}$$

\square

The second interpretation is in terms of the coloring of the hypercube $\{-1, 1\}^n$. It says that

$$\mathbf{Inf}[f] = \frac{\# \text{ of sensitive edges in the coloring defined by } f}{2^{n-1}} \quad (3)$$

This follows immediately from Fact 7.

The third interpretation is in terms of the distribution defined by the Fourier expansion of f . Recall that by Parseval's identity we have $\sum_{S \subseteq [n]} \widehat{f}(S)^2 = 1$ for any Boolean function f . Thus, the Fourier coefficients of f squared naturally define a probability distribution over subsets of $[n]$, denoted by \mathcal{S}_f , in which set S is sampled with probability $\widehat{f}(S)^2$. With that interpretation in mind, the total influence is just the expected size of $S \sim \mathcal{S}_f$.

Lemma 19. *For $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ we have*

$$\mathbf{I}[f] = \sum_{S \subseteq [n]} \widehat{f}(S)^2 \cdot |S|. \quad (4)$$

In particular, if f is Boolean-valued, then

$$\mathbf{I}[f] = \mathbf{E}_{\mathbf{S} \sim \mathcal{S}_f} [|\mathbf{S}|]. \quad (5)$$

Proof.

$$\mathbf{I}[f] = \sum_{i=1}^n \mathbf{Inf}_i[f] = \sum_{i=1}^n \sum_{S \ni i} \widehat{f}(S)^2 = \sum_{S \subseteq [n]} \widehat{f}(S)^2 \cdot |S|. \quad \square$$

Remark 20. This interpretation is nice since it tells us that if the influence is small, then most of the Fourier mass (i.e., the probability mass under the distribution \mathcal{S}_f) sits on the lower degree part of f . This is attained by a simple Markov's inequality. We will later show that several interesting classes of Boolean function have small total-influence, such as shallow decision trees, polynomial size CNF/DNF formulas, and more generally polynomial size constant-depth circuits (\mathbf{AC}^0).

We end this subsection with two questions that demonstrates the importance of the Parity and Majority functions.

Q1: Which n -variate Boolean function maximizes $\mathbf{I}[f]$? Of course the answer is Parity on n -variables whose influence is exactly n . Note however that $\mathbf{Eff}[\chi_{[n]}] = 0$.

Q2: Which n -variate Boolean function maximizes $\mathbf{Eff}[f]$? We claim that the answer is Majority.

Claim 21. *Let n be an odd number. Among all Boolean function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$, the Majority function is the unique maximizer of $\mathbf{Eff}[f] = \sum_{i=1}^n \widehat{f}(\{i\})$.*

Proof. Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be some Boolean function. We will show that $\mathbf{Eff}[f] \leq \mathbf{Eff}[\text{MAJ}]$ with equality holding only if $f = \text{MAJ}$.

$$\mathbf{Eff}[f] = \sum_{i=1}^n \widehat{f}(\{i\}) = \sum_{i=1}^n \mathbf{E}_{\mathbf{x}} [f(\mathbf{x}) \cdot \mathbf{x}_i] = \mathbf{E}_{\mathbf{x}} \left[f(\mathbf{x}) \cdot \sum_{i=1}^n \mathbf{x}_i \right]$$

Now observe that in order to maximize the right hand side we have to pick $f(x) = \text{sgn}(\sum_{i=1}^n x_i)$ for every input x . Thus, the right hand side is uniquely maximized by the Majority function. \square

The take-away message from the previous claim is that the total-effect cannot be larger than \sqrt{n} . In particular, if the function f has all effects the same, then all of them are smaller than $1/\sqrt{n}$. This intuitively shows that if all voters are equal, then their individual effect cannot be too large.

Remark 22. Since $\mathbf{Eff}[f] \leq \mathbf{Inf}[f]$, more functions have small total-effect. In particular, the Parity function has very small (in fact, 0) total-effect. More generally, polynomial size constant-depth circuits composed of AND, OR, NOT and Parity gates (the class called $\mathbf{AC}^0[\oplus]$) have small total-effect. It is an important open question to determine if this also the case for the class \mathbf{ACC}^0 – functions computed by polynomial size constant-depth circuits composed of AND, OR, NOT, and mod m gates, for any constant m .

Conjecture 23. For any $f \in \mathbf{ACC}^0$, $\mathbf{Eff}[f] \leq \text{polylog}(n)$.

If true, the conjecture would show that $\text{MAJ} \notin \mathbf{ACC}^0$, which is a longstanding conjecture by Roman Smolensky from the 80s. In particular, it would show that $\mathbf{TC}^0 \not\subseteq \mathbf{ACC}^0$ whereas currently we don't even know if $\mathbf{NP} \not\subseteq \mathbf{ACC}^0$. (In a recent breakthrough, Williams and Murray showed that \mathbf{NQP} or non-deterministic quasi-polynomial time cannot be computed by polynomial size \mathbf{ACC}^0 circuits). In general, a natural question to ask is: which functions have small total effect?

2.3 Poincare Inequality

Lemma 24 (Poincare Inequality). For any $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$, $\mathbf{I}[f] \geq \mathbf{Var}[f]$.

Proof. $\mathbf{I}[f] = \sum_{S \subseteq [n]} |S| \cdot \widehat{f}(S)^2 = \sum_{\emptyset \neq S \subseteq [n]} |S| \cdot \widehat{f}(S)^2 \geq \sum_{\emptyset \neq S \subseteq [n]} \widehat{f}(S)^2 = \mathbf{Var}[f]$. □

The Poincare Inequality can be interpreted as an isoperimetric inequality connecting the size of a set $A \subseteq \{-1, 1\}^n$, with the number of edges between A and its complement \bar{A} .

Corollary 25. Let $G = (V, E)$ be the Boolean Hypercube graph, with $V = \{-1, 1\}^n$ and edges connect two vertices $x, y \in V$ iff x and y differ in exactly one coordinate. Let $A \subseteq \{-1, 1\}^n$ of size $\alpha \cdot 2^n$. Then

$$|E(A, \bar{A})| \geq 2\alpha(1 - \alpha) \cdot 2^n$$

where $E(A, \bar{A})$ is the set of edges in G going from A to \bar{A} .

Proof. Take $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ to be the indicator function of A . That is, $f(x) = -1$ if and only if $x \in A$. We get that $\mathbf{Var}[f] = 4 \cdot \alpha \cdot (1 - \alpha)$. Also, we get that

$$\mathbf{Inf}[f] = \frac{\# \text{ of sensitive edges for } f}{2^{n-1}} = \frac{|E(A, \bar{A})|}{2^{n-1}}.$$

By Poincare Inequality, we get $|E(A, \bar{A})| \geq 4\alpha(1 - \alpha) \cdot 2^{n-1}$. □

From its proof, it is apparent that Poincare's inequality is not always tight. In fact, it is only tight for the constant functions or the dictators. A better isoperimetric inequality was given in 1964 by Harper.

Theorem 26. Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ with $\alpha = \mathbf{Pr}_{\mathbf{x}}[f(\mathbf{x}) = -1]$. Then, $\mathbf{I}[f] \geq 2\alpha \log(1/\alpha)$.

We will skip the proof (at least for now). Observe that for $\alpha = 2^{-k}$ this inequality is tight, as observed by the function $f(x) = -1$ iff $x_1 = x_2 = \dots = x_k = -1$. (In other words, this is the indicator function of the $n - k$ dimensional subcube defined by $x_1 = x_2 = \dots = x_k = -1$)

3 Noise Stability

Noise Stability captures the probability that the function-value remains the same when we perturb the input, flipping each bit with some fixed small probability δ . Informally, functions for which the function-value changes with small probability under the perturbation are called noise stable. The Dictator and the Majority functions are noise stable. On the other extreme, functions whose value changes dramatically under small perturbations are called noise sensitive. The parity function is such a function.

Definition 27. Let $\rho \in [-1, 1]$. For a fixed $x \in \{-1, 1\}^n$ we write $\mathbf{y} \sim N_\rho(x)$ to denote the random string such that for each $i \in [n]$ independently

$$\mathbf{y}_i = \begin{cases} x_i, & \text{with probability } \frac{1+\rho}{2}. \\ -x_i, & \text{with probability } \frac{1-\rho}{2}. \end{cases}$$

We say that \mathbf{y} is ρ -correlated with x .

Observe that if $\mathbf{y} \sim N_\rho(x)$ then for all $i \in [n]$, we have $\mathbf{E}[x_i \mathbf{y}_i] = \rho$.

Definition 28. A ρ -correlated pair (\mathbf{x}, \mathbf{y}) is sampled by first drawing $\mathbf{x} \sim \{-1, 1\}^n$ uniformly at random, and then drawing $\mathbf{y} \sim N_\rho(\mathbf{x})$. Alternatively, for each $i \in [n]$ independently, we draw $(\mathbf{x}_i, \mathbf{y}_i) \in \{-1, 1\}^2$ such that $\mathbf{E}[\mathbf{x}_i] = \mathbf{E}[\mathbf{y}_i] = 0$ and $\mathbf{E}[\mathbf{x}_i \mathbf{y}_i] = \rho$.

We are ready to define the *Noise-Stability* of a Boolean function.

Definition 29 (Noise-Stability). Let $f : \{-1, 1\}^n \rightarrow \mathbb{R}$. Then the noise-stability of f at ρ is

$$\mathbf{Stab}_\rho[f] = \mathbf{E}_{(\mathbf{x}, \mathbf{y}) \text{ } \rho\text{-corr.}} [f(\mathbf{x}) \cdot f(\mathbf{y})].$$

We observe that if f is Boolean-valued

$$\mathbf{Stab}_\rho[f] = 2 \cdot \mathbf{Pr}_{(\mathbf{x}, \mathbf{y}) \text{ } \rho\text{-corr.}} [f(\mathbf{x}) = f(\mathbf{y})] - 1.$$

Examples:

1. The Dictator function. $\mathbf{Stab}_\rho[\chi_i] = \rho$.
2. The Parity function. $\mathbf{Stab}_\rho[\chi_{[n]}] = \mathbf{E}[\prod_{i=1}^n \mathbf{x}_i \mathbf{y}_i] = \prod_{i=1}^n \mathbf{E}[\mathbf{x}_i \mathbf{y}_i] = \rho^n$.
3. The Stability of Majority is more complicated to analyze and we defer the proof for later on

Theorem 30.

$$\lim_{n \rightarrow \infty} \mathbf{Stab}_\rho[\text{MAJ}_n] = \frac{2}{\pi} \arcsin(\rho).$$

In particular, when ρ is close to 0, $\mathbf{Stab}_\rho[\text{MAJ}]$ behaves like $\frac{2}{\pi}\rho$, and when ρ is close to 1, then $\mathbf{Stab}_\rho[\text{MAJ}]$ behaves like $1 - O(\sqrt{1 - \rho})$.

To focus on the case that ρ is close to 1 we have the following definition.

Definition 31 (Noise Sensitivity). Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$, and $\delta \in [0, 1]$. Sample $\mathbf{x} \sim \{-1, 1\}^n$ and pick \mathbf{y} by flipping each bit of \mathbf{x} independently with probability δ . Then the noise sensitivity is defined to be

$$\mathbf{NS}_\delta[f] = \Pr[f(\mathbf{x}) \neq f(\mathbf{y})].$$

Alternatively,

$$\mathbf{NS}_\delta[f] = \frac{1}{2} - \frac{1}{2} \mathbf{Stab}_{1-2\delta}[f].$$

Theorem 30 implies that $\mathbf{NS}_\delta[\text{MAJ}] = O(\sqrt{\delta})$. In fact, we will see that this is the case for any LTF. Intuitively, this means that changing a small δ fraction of the input bits changes the value of the majority function with small $O(\sqrt{\delta})$ probability.

3.1 Fourier Expression for Stability

We define a very important linear operator on functions called the noise operator.

Definition 32. The noise operator $T_\rho(x)$ with parameter $\rho \in [-1, 1]$ maps $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ to $T_\rho f : \{-1, 1\}^n \rightarrow \mathbb{R}$ by

$$T_\rho f(x) = \mathbf{E}_{\mathbf{y} \sim N_\rho(x)} [f(\mathbf{y})].$$

Claim 33. $T_\rho f = \sum_{S \subseteq [n]} \widehat{f}(S) \cdot \rho^{|S|} \cdot \chi_S$.

Proof. Similarly to the proof of Fact 13 concerning the Fourier expansion of $D_i f$, we first observe that T_ρ is a linear operator. Thus, it suffices to check that $T_\rho \chi_S = \rho^{|S|} \chi_S$. Indeed,

$$T_\rho \chi_S(x) = \mathbf{E}_{\mathbf{y} \sim N_\rho(x)} \left[\prod_{i \in S} \mathbf{y}_i \right] = \prod_{i \in S} \mathbf{E}[\mathbf{y}_i] = \prod_{i \in S} (\rho \cdot x_i) = \rho^{|S|} \cdot \chi_S(x). \quad \square$$

We get

$$\mathbf{Stab}_\rho[f] = \mathbf{E}_{x \sim \{-1, 1\}^n} [f(x) T_\rho f(x)] = \langle f, T_\rho f \rangle = \sum_{S \subseteq [n]} \widehat{f}(S) \widehat{T_\rho f}(S) = \sum_{S \subseteq [n]} \widehat{f}(S)^2 \cdot \rho^{|S|} \quad (6)$$

What's the most stable function? The constant function. What if we look only on functions with $\Pr[f(\mathbf{x}) = 1] = 1/2$, i.e. unbiased function? Then, the dictator is the most stable. Indeed, $\mathbf{Stab}_\rho[\chi_i] = \rho$ and this is the largest stability among unbiased functions

Fact 34. If $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ is unbiased, i.e., $\widehat{f}(\emptyset) = 0$, then

$$\mathbf{Stab}_\rho[f] = \sum_{S \subseteq [n]} \widehat{f}(S)^2 \rho^{|S|} \underset{\widehat{f}(\emptyset)=0}{=} \sum_{\emptyset \neq S \subseteq [n]} \widehat{f}(S)^2 \rho^{|S|} \leq \rho \cdot \sum_{\emptyset \neq S \subseteq [n]} \widehat{f}(S)^2 \leq \rho.$$

Notice that the stability equals ρ iff all the Fourier mass is on degree-1 coefficients. The next claim shows that such functions must be dictators (or anti-dictators).

Claim 35. Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ with $\widehat{f}(S) = 0$ for all $|S| > 1$. Then, f is either constant, dictator or anti-dictator (the anti-dictator is the function $-\chi_i$).

Proof. We can write $f(x) = a_0 + \sum_{i=1}^n a_i x_i$ where $a_0 = \widehat{f}(\emptyset)$ and $a_i = \widehat{f}(\{i\})$. Suppose f is not a constant function. Then, there exists an i such that $a_i \neq 0$. Consider the function $D_i f(x)$. By the formula for derivatives $D_i f(x) = a_i$. In addition, since $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ is Boolean-valued $D_i f(x) \in \{-1, 0, 1\}$ and we get that $a_i \in \{-1, 0, 1\}$. But, since we assumed that $a_i \neq 0$ we get $a_i \in \{-1, 1\}$. Finally, observe that by Parseval's identity $a_0^2 + \sum_j a_j^2 = 1$, thus all other coefficients must be 0 and we get $f(x) = a_i x_i$ where a_i is either 1 or -1. \square

4 Arrow's Theorem

Suppose we have a 3-options a , b and c (e.g., Chocolate, Vanilla, or Pistachio). We have n voters, each has a ranking over the options $a > b > c$ or $c > b > a$, etc. There are $3! = 6$ such rankings. We would like to aggregate these rankings to a collective decisions while respecting:

1. Independence of Irrelevant Alternatives (IIA, in short): whether society prefers a over b does not depend on c .
2. Unanimous: If all voters prefer a over b , then society should prefer a over b .
3. Rationality: Society should not prefer a over b , b over c and c over a . In general, the directed graph of preferences should be a cycle-free.

Given rule 1, we see that we can think of an 3-way election as a three independent 2-way election between each of the pairs. This was suggested by Condorcet in the 18th century.

We those need to pick three Boolean function $f, g, h : \{-1, 1\}^n \rightarrow \{-1, 1\}$ (or voting rules), such that

	voter 1	voter 2	voter 3	...	voter n	Outcome
$x = a(+1)$ vs. $b(-1)$	+1	+1	+1	...	+1	$f(x)$
$y = b(+1)$ vs. $c(-1)$	+1	-1	-1	...	+1	$g(y)$
$z = c(+1)$ vs. $a(-1)$	-1	+1	-1	...	-1	$h(z)$

What would happen if all voting rules, f , g and h , were equal to MAJ_n ? Then, the result can be irrational (as observed by Condorcet). The simple example (called Condorcet's Paradox) is the following table:

	voter 1	voter 2	voter 3	Outcome
$x = a(+1)$ vs. $b(-1)$	+1	+1	-1	+1
$y = b(+1)$ vs. $c(-1)$	+1	-1	+1	+1
$z = c(+1)$ vs. $a(-1)$	-1	+1	+1	+1

In this setting, society prefers a over b , b over c , and c over a , which is irrational. Can this be avoided?

Theorem 36 (Arrow's Theorem). *The only voting rule that satisfies all three properties is the Dictatorship, i.e., when $f = g = h = \chi_i$ for some $i \in [n]$.*

We will prove this under the assumption that $f = g = h$.

4.1 Kalai's Proof of Arrow's Theorem

Note that each voter gives one of the $3! = 6$ rankings over a , b , and c , which is captured by 6 out of the 8 vectors in $\{-1, 1\}^3$. The only vectors that do not correspond to rational rankings are $(+1, +1, +1)$ and $(-1, -1, -1)$. That is, rational rankings are exactly the strings that satisfy the Not-All-Equal predicate $\text{NAE}_3 : \{-1, 1\}^3 \rightarrow \{0, 1\}$.

Definition 37. *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$. Let $x, y, z \in \{-1, 1\}^n$, s.t., $\forall i \in [n]$ we have $\text{NAE}(x_i, y_i, z_i) = 1$. Then, the outcome $(f(x), f(y), f(z))$ is rational if $\text{NAE}(f(x), f(y), f(z)) = 1$.*

Theorem 38 (Kalai'02). *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be the voting rule. Assuming each voter ranking is independently uniform over all 6 possibilities. Then*

$$\Pr_{\mathbf{x}, \mathbf{y}, \mathbf{z}} [(f(\mathbf{x}), f(\mathbf{y}), f(\mathbf{z})) \text{ is rational}] = \frac{3}{4} - \frac{3}{4} \mathbf{Stab}_{(-1/3)}[f]. \quad (7)$$

Furthermore, if the output is always rational, and f is unanimous, then f must be a dictator.

Proof. Let $\text{NAE}_3 : \{-1, 1\}^3 \rightarrow \{0, 1\}$ be the Not-All-Equal predicate. We have

$$\text{NAE}_3(w_1, w_2, w_3) = \frac{3}{4} - \frac{1}{4}w_1w_2 - \frac{1}{4}w_1w_3 - \frac{1}{4}w_2w_3.$$

Thus, by linearity of expectation

$$\Pr[(f(\mathbf{x}), f(\mathbf{y}), f(\mathbf{z})) \text{ is rational}] = \frac{3}{4} - \frac{1}{4} \mathbf{E}[f(\mathbf{x})f(\mathbf{y})] - \frac{1}{4} \mathbf{E}[f(\mathbf{x})f(\mathbf{z})] - \frac{1}{4} \mathbf{E}[f(\mathbf{y})f(\mathbf{z})]$$

By symmetry all these three expectations are equal, thus

$$\Pr[(f(\mathbf{x}), f(\mathbf{y}), f(\mathbf{z})) \text{ is rational}] = \frac{3}{4} - \frac{3}{4} \mathbf{E}[f(\mathbf{x})f(\mathbf{y})]$$

and we are left to prove that $\mathbf{E}[f(\mathbf{x})f(\mathbf{y})] = \mathbf{Stab}_{(-1/3)}[f]$.

Note that the pairs $\{(\mathbf{x}_i, \mathbf{y}_i)\}$ are independent of one another. In addition, we see that $\mathbf{E}[\mathbf{x}_i] = \mathbf{E}[\mathbf{y}_i] = 0$ and that $\mathbf{E}[\mathbf{x}_i\mathbf{y}_i] = \frac{2}{6} \cdot (+1) + \frac{4}{6} \cdot (-1) = -1/3$. We get that \mathbf{x} and \mathbf{y} are a $(-1/3)$ -correlated pair. Thus, $\mathbf{Stab}_{(-1/3)}[f] = \mathbf{E}[f(\mathbf{x})f(\mathbf{y})]$ which completes the proof of Eq. (7).

To show the furthermore part, observe that the probability of a rational outcome equals 1 iff the stability $\mathbf{Stab}_{(-1/3)}[f] = -1/3$. Equivalently, $-\mathbf{Stab}_{(-1/3)}[f] = 1/3$. But observe that

$$\begin{aligned} -\mathbf{Stab}_{(-1/3)}[f] &= - \sum_{S \subseteq [n]} \widehat{f}(S)^2 \cdot (-1/3)^{|S|} \\ &\leq \sum_{|S| \text{ odd}} \widehat{f}(S)^2 \cdot (1/3)^{|S|} \\ &\leq \frac{1}{3} \cdot \sum_{S: |S|=1} \widehat{f}(S)^2 + \frac{1}{27} \cdot \sum_{S: |S|>1, |S| \text{ odd}} \widehat{f}(S)^2 \end{aligned}$$

and the only way this will be equal to $1/3$ is if all the Fourier mass is on sets of size 1. In this case, we get that $\widehat{f}(S) = 0$ for all $|S| > 1$. Then, by claim 35, we see that f is either the dictator or the anti-dictator (constant functions do not have Fourier mass on sets of size 1, and therefore ruled out). The anti-dictator is not unanimous, thus, the only voting rule left is the dictator $f(x) = x_i$ for some $i \in [n]$. \square

Important Remark - Robustness of Kalai's Proof: An important remark is that Kalai's Proof actually can be extended to show that if $\Pr[(f(\mathbf{x}), f(\mathbf{y}), f(\mathbf{z})) \text{ is rational}] \geq 1 - \varepsilon$ for some small $\varepsilon > 0$ then f is $1 - O(\varepsilon)$ correlated with a Dictator/Anti-Dictator function. To show that you can notice that in this case, Equation (7) implies that $\sum_{S: |S|=1} \widehat{f}(S)^2 \geq 1 - \frac{9}{2} \cdot \varepsilon$ (we leave it as an Exercise). Finally, a theorem by Friedgut, Kalai, and Naor (FKN's theorem, in short) proves a robust version of Claim 35, showing that if $\sum_{S: |S|=1} \widehat{f}(S)^2 \geq 1 - \delta$ then there exists a coordinate i such that $|\langle f, \chi_i \rangle| \geq 1 - C \cdot \delta$, for some universal constant C (to be explicit, $C = 3202$, but this can be probably improved). We will later see the proof of FKN's theorem.