CS 294-92 Analysis of Boolean Functions
Problem Set 1

Due: February 10, 2020, 7 PM

Submission on gradescope. You are encouraged to discuss the problems and solve them in groups. However, the solutions are to be written up alone, listing all the collaborators.

1. Let $f : \{-1,1\}^n \to \mathbb{R}$. For each of the following functions g, express the Fourier expansion of g in terms of the Fourier expansion of f. That is, for $S \subseteq [n]$, write an expression for $\widehat{g}(S)$ in terms of the Fourier coefficients of f.

   (a) $g(x) = f(x) \cdot \prod_{i=1}^{n} x_i$.

   (b) $g(x) = \frac{1}{2}(f(x) - f(-x))$. (This function is also called f^{odd}. Do you see why?)

   (c) $g(x) = \frac{1}{2}(f(x) + f(-x))$. (This function is also called f^{even}.)

   (d) $g(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for some permutation $\pi : [n] \to [n]$.

   (e) $g(x) = f(x \odot a)$ for some fixed $a \in \{-1,1\}^n$ (where $x \odot a$ means point-wise product.)

   (f) $g(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = f(x_1, \ldots, x_k, 1, 1, \ldots, 1)$ for some $0 \le k \le n$.

   (g) $g(x_1, \ldots, x_k) = f(x_1, \ldots, x_k, 1, 1, \ldots, 1)$ for some $0 \le k \le n$.

2. Compute the Fourier expansion of the following functions:

   (a) $\min_2 : \{-1,1\}^2 \to \{-1,1\}$, the minimum function on 2 bits.

   (b) $\mathrm{MUX}_3 : \{-1,1\}^3 \to \{-1,1\}$ which outputs x_2 if $x_1 = 1$ and x_3 if $x_1 = -1$.

   (c) The Not-All-Equal function $\mathrm{NAE}_3 : \{-1,1\}^n \to \{0,1\}$ defined by $\mathrm{NAE}_3(x) = 1$ iff the bits x_1, x_2, x_3 are not all equal.

   (d) The Inner-Product mod 2 function $\mathrm{IP}_{2n} : \mathbb{F}_2^{2n} \to \{-1,1\}$, defined by

   $$\mathrm{IP}_{2n}(x_1, \ldots, x_n, y_1, \ldots, y_n) = (-1)^{\sum_{i=1}^{n} x_i y_i}.$$

3. (Ex. 1.4, O'Donnell's book) Let $f : \{-1,1\}^n \to \mathbb{R}$ have Fourier expansion

   $$f(x) = \sum_{S \subseteq [n]} \widehat{f}(S) \prod_{i \in S} x_i.$$

   Let $F : \mathbb{R}^n \to \mathbb{R}$ be the (multilinear) extension of f which is also defined by

   $$F(x) = \sum_{S \subseteq [n]} \widehat{f}(S) \prod_{i \in S} x_i.$$

   Show that if $\mu = (\mu_1, \ldots, \mu_n) \in [-1,1]^n$, then $F(\mu) = \mathbf{E}_{\mathbf{y}}[f(\mathbf{y})]$ where $\mathbf{y}$ is the random string in $\{-1,1\}^n$ defined having $\mathbf{E}[\mathbf{y}_i] = \mu_i$ independently for all $i \in [n]$.

4. For $n \in \mathbb{N}$, let H_{2^n} be the 2^n-by-2^n Hadamard matrix defined as follows. We identify the row indices with $\mathbb{F}_2^n$ and similarly for the columns. The (γ, x) entry of H_{2^n}, for $\gamma \in \mathbb{F}_2^n$ and $x \in \mathbb{F}_2^n$ is defined to be $(-1)^{\sum_{i=1}^n \gamma_i \cdot x_i}$.

   (a) Let $f : \mathbb{F}_2^n \to \mathbb{R}$. Identify the truth-table f with a column vector $v \in \mathbb{R}^{2^n}$, satisfying $v_x = f(x)$ for $x \in \mathbb{F}_2^n$. Take u to be the column vector defined by $u = 2^{-n} \cdot H_{2^n} \cdot v$. Show that for any $\gamma \in \mathbb{F}_2^n$:
   $$u_\gamma = \widehat{f}(\{i \in [n] : \gamma_i = 1\}).$$
   That is, u is the vector of Fourier coefficients of f.

   (b) Show that $H_{2^n} \cdot H_{2^n} = 2^n \cdot I$ (where I is the 2^n-by-2^n identity matrix) and deduce that $v = H_{2^n} \cdot u$. Put differently, this shows that the Fourier transform is an involution up to normalization, i.e.,
   $$\widehat{\widehat{f}} = 2^{-n} \cdot f.$$

5. In this question we will explore the representation of a Boolean function as a multilinear polynomial over $\{0,1\}^n$, express the connection between this representation and the Fourier representation and deduce results on the Fourier-sparsity of low-degree Boolean functions.

   (a) Let $F : \{0,1\}^n \to \mathbb{R}$. Show that F can be represented as a multilinear polynomial $F(x) = \sum_{S \subseteq [n]} c_S \cdot \prod_{i \in S} x_i$ over $\{0,1\}^n$ (where we think of 0 and 1 as real numbers). Hint: try writing first for any $a \in \{0,1\}^n$, the interpolating polynomial of the indicator function
   $$1_a(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise.} \end{cases}$$

   (b) Suppose $F : \{0,1\}^n \to \mathbb{Z}$. Show that all the coefficients c_S above are integers.

   (c) Let $F : \{0,1\}^n \to \mathbb{R}$ with multilinear expansion $F(x) = \sum_{S \subseteq [n]} c_S \cdot \prod_{i \in S} x_i$ over $\{0,1\}^n$, as in Item (5a). Let $f : \{-1,1\}^n \to \mathbb{R}$ defined by $f(x_1, \ldots, x_n) = F\left(\frac{1-x_1}{2}, \ldots, \frac{1-x_n}{2}\right)$. Show that
   $$\widehat{f}(T) = (-1)^{|T|} \cdot \sum_{S \supseteq T} 2^{-|S|} \cdot c_S$$
   for all $T \subseteq [n]$.
   Furthermore, letting $\deg(f)$ denote the total degree of f as a multilinear polynomial (and similarly $\deg(F)$), show that $\deg(f) = \deg(F)$.

   (d) Let $f : \{-1,1\}^n \to \mathbb{Z}$. Show that if $\deg(f) = k$, then every Fourier coefficient of f is an integer multiple of 2^{-k}.

   (e) Deduce that if $f : \{-1,1\}^n \to \{-1,1\}$ has degree k, then it has at most 4^k non-zero Fourier coefficients.