CS 294-92 Analysis of Boolean Functions
Problem Set 2

Due: Feb 28, 2020, 7 PM

You are encouraged to discuss the problems and solve them in groups. However, the solutions are to be written up alone, listing all the collaborators.

1. **The Gravity Operator:** The "gravity operator" G_i in direction i maps Boolean functions $f : \{-1,1\}^n \to \{-1,1\}$ to $G_i f : \{-1,1\}^n \to \{-1,1\}$ as follows.

$$G_i f(x) = \begin{cases} \mathtt{max}\{f(x^{i\mapsto -1}), f(x^{i\mapsto +1})\}, & \text{if } x_i = +1 \\ \mathtt{min}\{f(x^{i\mapsto -1}), f(x^{i\mapsto +1})\}, & \text{if } x_i = -1 \end{cases}$$

Observe that by definition, $G_i f$ is monotone in the i-th direction.

(a) Show that if f is monotone in the j-th direction then so is $G_i f$, for any $i, j \in [n]$.

(b) Show that $\mathbf{Pr}_{\mathbf{x}\sim\{-1,1\}^n}[f(\mathbf{x}) = 1] = \mathbf{Pr}_{\mathbf{x}\sim\{-1,1\}^n}[G_i f(\mathbf{x}) = 1]$.

(c) Show that $\mathbf{Inf}_i[f] = \mathbf{Inf}_i[G_i f]$ and $\mathbf{Inf}_j[G_i f] \le \mathbf{Inf}_j[f]$ for $j \ne i$.

(d) An edge-isoperimetric inequality on functions $f : \{-1,1\}^n \to \{-1,1\}$ is an inequality of the form $\mathbf{I}[f] \ge \phi(\mathbf{Pr}[f(\mathbf{x}) = 1])$ for some function $\phi : [0,1] \to \mathbb{R}$ (e.g., Poincare's Inequality correspond to $\phi(\alpha) = 4\alpha(1-\alpha)$, and Harper's Inequality correspond to $\phi(\alpha) = 2\alpha\log_2(1/\alpha)$). Deduce that it suffices to prove edge-isoperimetric inequalities for monotone functions.

2. **Stability of Arrow's Theorem & Extensions:** Recall that we proved in class Kalai's Theorem stating that

$$\mathop{\mathbf{Pr}}_{\mathbf{x},\mathbf{y},\mathbf{z}}[(f(\mathbf{x}), f(\mathbf{y}), f(\mathbf{z})) \text{ is rational}] = \tfrac{3}{4} - \tfrac{3}{4}\mathbf{Stab}_{(-1/3)}[f]. \tag{1}$$

where for each $i \in [n]$ independently $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ are selected uniformly at random from all strings in $\{-1,1\}^3$ that are not-all-equal, and a result is called rational if the three bits are not-all equal.

(a) Suppose $1 - \varepsilon = \mathbf{Pr}_{\mathbf{x},\mathbf{y},\mathbf{z}}[(f(\mathbf{x}), f(\mathbf{y}), f(\mathbf{z})) \text{ is rational}]$. Show that $\mathbf{W}^1[f] \ge 1 - 9\varepsilon/2$.

(b) (Bonus) Extend the previous item to the case of three Boolean functions, $f_1, f_2, f_3 : \{-1,1\}^n \to \{-1,1\}$ as long as $\mathbf{E}[f_1]\,\mathbf{E}[f_2] + \mathbf{E}[f_2]\,\mathbf{E}[f_3] + \mathbf{E}[f_1]\,\mathbf{E}[f_3] \ge 0$. Namely, show that if

$$1 - \varepsilon = \mathop{\mathbf{Pr}}_{\mathbf{x},\mathbf{y},\mathbf{z}}[(f_1(\mathbf{x}), f_2(\mathbf{y}), f_3(\mathbf{z})) \text{ is rational}]$$

then $\mathbf{W}^1[f_1], \mathbf{W}^1[f_2], \mathbf{W}^1[f_3] \ge 1 - O(\varepsilon)$.

(c) Show that there exist functions f_1, f_2, f_3 that are not dictators nor anti-dictators s.t.

$$\mathop{\mathbf{Pr}}_{\mathbf{x},\mathbf{y},\mathbf{z}}[(f_1(\mathbf{x}), f_2(\mathbf{y}), f_3(\mathbf{z})) \text{ is rational}] = 1.$$

3. **Influences in Elections:**

   (a) Ex. 2.9 in O'Donnell's Book - "In 1965, the Nassau County (New York) Board used a weighted majority voting system to make its decisions, with the 6 towns getting differing weights based on their population. Specifically, the board used the voting rule $f : \{-1,1\}^6 \to \{-1,1\}$ given by the weighted majority

   $$f(x) = \operatorname{sgn}(31x_1 + 31x_2 + 28x_3 + 21x_4 + 2x_5 + 2x_6).$$

   Compute $\mathbf{Inf}_i[f]$ for all $i \in [6]$."

   (b) We define the composition of two Boolean functions $f : \{-1,1\}^n \to \{-1,1\}$ and $g : \{-1,1\}^m \to \{-1,1\}$, denote by $f \circ g : \{-1,1\}^{mn} \to \{-1,1\}$, as

   $$(f \circ g)(x) = f(g(x_1, \ldots, x_m), g(x_{m+1}, \ldots, x_{2m}), \ldots, g(x_{m(n-1)+1,}, \ldots, x_{mn})).$$

   What are the influences of the function $\mathrm{MAJ}_n \circ \mathrm{MAJ}_n$? (asymptotically)
   What are the influences of the function $\underbrace{\mathrm{MAJ}_3 \circ \mathrm{MAJ}_3 \circ \ldots \circ \mathrm{MAJ}_3}_{k \text{ times}}$? (exactly)

   (c) (Bonus) Suppose you are a citizen in Randomland. Elections in Randomland are run between the two candidates: Positive and Negative. There are n citizens in Randomland. Out of them m citizens have decided to form the state of North-Random, and the others are "loners". Assume m and n are odd. Elections are run as follows: everyone votes, but decide their vote uniformly at random. The majority winner in the state of North-Random gets m votes. Remaining votes by "loners" are counted directly. Then, based on the overall count the winner is decided. Estimate asymptotically in terms of m and n the influence of a loner versus the influence of a North-Random resident.

4. **Fourier Concentration of Decision Trees:** Let $f : \{-1,1\}^n \to \{-1,1\}$ be computable by a decision tree of size s and let $\varepsilon \in (0,1]$. Show that the spectrum of f is 4ε-concentrated up to degree $\log_2(s/\varepsilon)$.

5. **Lipschitz functions and Poincare's Inequality.**

   (a) We say that a function $f : \{-1,1\}^n \to \mathbb{R}$ is L-Lipschitz if for every neighboring points (x, y) on the hypercube $\{-1,1\}^n$, we have $|f(x) - f(y)| \le L$. Show that if f is L-Lipschitz then

   $$\mathop{\mathbf{E}}_{\mathbf{x}\sim\{\pm1\}^n} \mathop{\mathbf{E}}_{\mathbf{y}\sim\{\pm1\}^n} [(f(\mathbf{x}) - f(\mathbf{y}))^2] \le n \cdot L^2/2.$$

   (Hint: Use Poincare's Inequality).

   (b) **Extension to higher dimensions:** Suppose $f : \{-1,1\}^n \to \mathbb{R}^d$ is a function such that $\|f(x) - f(y)\|_2 \le L$ for all neighboring points (x, y) on the hypercube (where $\|v\|_2 = \sqrt{\sum_{i=1}^d v_i^2}$ for $v \in \mathbb{R}^d$). Show that

   $$\mathop{\mathbf{E}}_{\mathbf{x}\sim\{\pm1\}^n} \mathop{\mathbf{E}}_{\mathbf{y}\sim\{\pm1\}^n} \left[\|f(\mathbf{x}) - f(\mathbf{y})\|_2^2\right] \le n \cdot L^2/2.$$

   (c) Let $DTSize : \{-1,1\}^{2^n} \to \mathbb{N}$ be the function that on a given input $x \in \{-1,1\}^{2^n}$, interprets x as a truth-table of a function $f_x : \{-1,1\}^n \to \{-1,1\}$, and outputs the size of the smallest decision tree computing f_x. Show that $DTSize$ is n-Lipschitz.

   (d) Deduce that there exists a value s such that most (say 0.99 fraction) of the functions on n variables have decision tree size in the range $[s, s + O(2^{n/2} \cdot n)]$. (Note: You don't need to determine s, just prove that it exists).